A CLASS OF SELF-ADJOINT BOUNDARY-VALUE PROBLEMS FOR SYSTEMS OF FIRST ORDER LINEAR DIFFERENTIAL EQUATIONS

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TABLE OF CONTENTS

SUMMARY

The detailed theory of eigenvalue problems involving an n-th order linear scalar differential equation is very well known. The purpose of this study is to extend this theory to a special class of systems of linear first order differential equations.

The eigenvalue problem studied here can be described as follows: Let $P_o(t)$ and $P_1(t)$ be two r-by-r matrices defined on the real finite interval $a \leq t \leq b$. Suppose that $P_o(t)$, $P_o'(t)$ and $P_1(t)$ are continuous and that $P_o(t)$ is nonsingular, $a \leq t \leq b$. Moreover, let M and N be constant r-by-r matrices. For a continuously differentiable vector $x(t)$, $a \leq t \leq b$, having r components, define the linear operators $\mathcal{L}$ and $\mathcal{B}$ by the relations

$$\mathcal{L}\, x(t) = P_o(t)x'(t) + P_1(t)x(t), \; a \leq t \leq b,$$

$$\mathcal{B}\, x = Mx(a) + Nx(b).$$

The eigenvalue problem consists of finding a continuously differentiable vector $\phi(t)$ which does not vanish identically on $a \leq t \leq b$, and a (possibly complex) constant ℓ so that both

$$\mathcal{L}\phi(t) = \ell\phi(t), \; a \leq t \leq b,$$

and

$$\mathcal{B}\phi = 0$$

are satisfied. Such a function $\phi(t)$ is then called an eigenfunction and ℓ is called an eigenvalue.

Special attention is given to the self-adjoint eigenvalue problem, where for any two vectors u(t), v(t) with continuous derivatives and for which $\mathcal{B}\, u = \mathcal{B}\, v = 0$

$$\int_a^b v^* \mathcal{L}\, u \, dt = \int_a^b (\mathcal{L}\, v)^* u \, dt$$

holds. In the above relation v* stands for the conjugate transpose of the one-column matrix v. It is shown that a self-adjoint eigenvalue problem always has at most a countably infinite number of real eigenvalues, some of which can be multiple. From the corresponding eigenfunctions a complete orthonormal system is constructed in the space $(L_2[a,b])^r$, the space generated by forming the Cartesian product of r factors each being the space $L_2[a,b]$.

In case of ℓ not an eigenvalue a solution of the boundary value problem

$$\mathcal{L}\, x(t) = \ell x(t) + f(t), \quad a \leq t \leq b,$$

$$\mathcal{B}\, x = 0,$$

where f(t) is a vector continuous on $a \leq t \leq b$, exists and is expressed by means of the Green's matrix associated with the eigenvalue problem.

In the final chapter two examples are given.

CHAPTER I

INTRODUCTION

Let $P_0(t)$ and $P_1(t)$ be two continuous r-by-r matrices of possibly complex valued functions on the interval $a \leq t \leq b$. It will be assumed throughout that $P_0(t)$ is nonsingular and that $P_0'(t)$, the derivative of $P_0(t)$, is continuous on $a \leq t \leq b$.

Let the linear operator $\mathcal{L}$ be defined by the identity

$$\mathcal{L}\, x(t) = P_0(t)x'(t) + P_1(t)x(t), \; a \leq t \leq b, \tag{1}$$

where $x(t)$ is an r-dimensional vector continuously differentiable on $a \leq t \leq b$. Define $\mathcal{B}$ by

$$\mathcal{B}x = Mx(a) + Nx(b) \tag{2}$$

where M and N are arbitrary constant r-by-r matrices, both of which are not zero matrices.

The principal objective of this investigation is to study the boundary-value problem

$$\pi: \quad \mathcal{L}\, x(t) = \ell x(t), \quad \mathcal{B}x = 0$$

where ℓ is a (possibly complex) parameter.

Multiplying the differential equation by $P_0^{-1}(t)$ on the left one notices that the problem π is equivalent to the problem

$$x' = A(t)x + \ell B(t)x, \quad \mathcal{B}x = 0,$$

where the matrices A(t), B(t) and B'(t) are continuous and B(t) is nonsingular. A more general equation where B(t) is just assumed to be continuous and not identically equal to the zero matrix is treated in [1]*, [2]. π is therefore a special case of that general problem. Another specialization is the boundary-value problem for a scalar function u(t)

$$a_o u^{(n)} + a_1 u^{(n-1)} + \ldots + a_{n-1} u' + a_n u = \ell u$$

with $a_o(t) \neq 0$ and appropriate boundary conditions. A theory of this eigenvalue problem is given in [3].

Definition 1: If for $\ell = \lambda$ the problem π has a nontrivial solution $\phi(t)$ (i.e., $\phi \not\equiv 0$ is a solution of $\mathcal{L}x = \lambda x$ on [a, b] and $\mathcal{B}\phi = 0$) then λ is called an eigenvalue of π and the vector $\phi(t)$ a corresponding eigenfunction.**

The main problem will be to demonstrate existence of eigenvalues of π and to study the properties of eigenvalues and eigenfunctions.

Of interest also will be the inhomogeneous problem

$$\mathcal{L}x = \ell x + f, \tag{3}$$

where f(t) is a given continuous vector.

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*Numbers in square brackets refer to references listed in the bibliography.

**Since the notion of eigenvector already has another meaning in literature, the word eigenfunction is used here.

To facilitate the exposition the notations as well as some elementary properties of matrices to be used throughout will be given.

If:

$$A(t) = (\alpha_{ij}(t)), \quad i = 1, \ldots, n, \quad j = 1, \ldots, m, \tag{4}$$

is a n-by-m matrix of differentiable functions, the derivative of $A(t)$ is the matrix

$$A'(t) = (\alpha_{ij}'(t)), \quad i = 1, \ldots, n, \quad j = 1, \ldots, m. \tag{5}$$

Similarly if $A(t)$ is a n-by-m matrix of integrable functions, the integral of $A(t)$ is the matrix

$$\int_{t_o}^{t} A(s)\, ds = \left(\int_{t_o}^{t} \alpha_{ij}(s)\, ds \right), \quad i = 1, \ldots, n \quad j = 1, \ldots, m. \tag{6}$$

Note that

$$\left(\int_{t_o}^{t} A(s)\, ds \right)' = A(t)$$

at each point of continuity of $A(t)$.

The above definitions of derivative and integral of matrices include in particular the cases of one-column matrices, i.e., vectors.

A number of properties of derivative and integral of a scalar function also hold for the case of matrix. The identities listed below are easy to verify.

1) If α and β are scalars, then

$$(\alpha A(t) + \beta B(t))' = \alpha A'(t) + \beta B'(t), \tag{7}$$

and also

$$\int_{t_o}^{t} (\alpha A(s) + \beta B(s))ds = \alpha \int_{t_o}^{t} A(s)ds + \beta \int_{t_o}^{t} B(s)ds.$$

2) If $A(t)$ and $B(t)$ are matrices, then

$$(A(t)\, B(t))' = A'(t)\, B(t) + A(t)\, B'(t). \tag{8}$$

3) If $A(t)$ is a matrix and if p and q are vectors independent of t, then

$$\int_{a}^{b} A(t)p\, dt = \left[\int_{a}^{b} A(t)\, dt \right] \cdot p\,, \tag{9}$$

$$\int_{a}^{b} q^* A(t)\, dt = q^* \int_{a}^{b} A(t)\, dt\,.$$

The identities (7, 8, 9) hold whenever the occurring expressions have a meaning.

Throughout this paper A^* stands for the conjugate transpose of the matrix A. In case of a scalar α the conjugate value may be denoted by $\bar{\alpha}$.

<u>Definition 2</u>: Suppose the components $u_{(i)}(t)$, $i = 1, 2, \ldots, r$, and $v_{(i)}(t)$, $i = 1, 2, \ldots, r$, of the r-dimensional vectors u, v are all elements of the space $L_2[a, b]$. Then u and v belong to the space $(L_2[a, b])^r$, the space generated by forming the Cartesian product of r factors, each identical to the space $L_2[a, b]$. The scalar product of u and v, denoted by (u, v), will be defined by the expression

$$(u, v) = \int_a^b v^* u \, dt = \tag{10}$$

$$= \sum_{i=1}^{r} \left(\int_a^b u_{(i)} \, \bar{v}_{(i)} \, dt \right) .$$

A number of properties of the scalar product follow directly from its definition. A consequence of the hypothesis is that

$$(u, u) < \infty, \; (v, v) < \infty \; .$$

Of basic significance are the following algebraic properties:

1) $(u, u) \geq 0$ (11)

for all $u \in (L_2[a, b])^r$.

2) $(u, u) = 0$, if and only if $u = 0$ a.e. on $[a, b]$. (12)

3) $(v, u) = \overline{(u, v)}$ (13)

for all $u, v \in (L_2[a, b])^r$.

4) If α and β are scalars and u_1, u_2 and v are vectors belonging to $(L_2[a, b])^r$, then

$$(\alpha u_1 + \beta u_2, v) = \alpha(u_1, v) + \beta(u_2, v). \tag{14}$$

Definition 3: If a vector u belongs to $(L_2[a, b])^r$, then its norm $\|u\|$ is equal to

$$\|u\| = (u, u)^{1/2}. \tag{15}$$

Lemma 1: (Schwarz inequality).

$$|(u, v)| \leq \|u\| \cdot \|v\| \tag{16}$$

for all u, v.

Proof: For all real λ

$$\left(\frac{u}{(u, v)} + \lambda v, \frac{u}{(u, v)} + \bar{\lambda} v \right)$$

is real and nonnegative. By the algebraic properties of the scalar product the above expression is equal to

$$\frac{\|u\|^2}{|(u, v)|^2} + \lambda^2\|v\| + \lambda(v, \frac{u}{(u, v)}) + \frac{1}{(u, v)} (u, \bar{\lambda} v) =$$

$$= \frac{\|u\|^2}{|(u, v)|^2} + 2\lambda + \lambda^2\|v\|^2 .$$

Hence the quadratic equation in λ

$$\frac{\|u\|^2}{|(u, v)|^2} + 2\lambda + \lambda^2\|v\| = 0$$

cannot have two different real solutions λ and therefore cannot have a positive discriminant. From

$$1 - \frac{\|u\|^2 \ \|v\|^2}{|(u, v)|^2} \leq 0$$

(16) follows proving the lemma.

Lemma 2: (Minkowsky inequality or triangle inequality.) For all u, v

$$\|u + v\| \leq \|u\| + \|v\| \ . \tag{17}$$

The proof is a consequence of the Schwarz inequality.

Definition 4: Two vectors u and v are called orthogonal if their scalar product vanishes.

Another notation will be frequently used in this thesis. The class $C^n[a, b]$ contains all functions whose first n derivatives are continuous everywhere on $[a, b]$. The class C^o is denoted by C. A matrix is said to belong to $C^n[a, b]$ whenever all its elements belong to $C^n[a, b]$.

Finally note that the hypotheses on $P_o(t)$, $P_1(t)$ guarantee the existence on $a \leq t \leq b$ of a fundamental matrix of solutions for $\mathcal{L} x = \ell x$ for every complex ℓ by standard theory. The issue is only whether solutions satisfying the boundary conditions are possible.

CHAPTER II

SELF-ADJOINTNESS OF THE EIGENVALUE PROBLEM

An important property of the operator $\mathcal{L}$ will now be derived. With each operator $\mathcal{L}$ as given by (1) associate the operator $\mathcal{L}^+$ defined by

$$\mathcal{L}^+ x = -(P_0^* x)' + P_1^* x . \tag{18}$$

For the operators $\mathcal{L}$ and $\mathcal{L}^+$ one has the following identity:

Lemma 3: (Lagrange). If $u(t)$ and $v(t)$ are arbitrary r-dimensional vectors ϵ $C^1[a, b]$, then

$$v^* \mathcal{L} u - (\mathcal{L}^+ v)^* u = (v^* P_0 u)' . \tag{19}$$

Proof: The left hand side of (19) is by (1, 8, 18)

$$v^* P_0 u' + v^* P_1 u + (P_0^{*\prime} v + P_0^* v')^* u - (P_1^* v)^* u.$$

Since

$$(A\ B)^* = B^* A^* ,$$

there results

$$v^* P_0 u' + v^* P_1 u + v^* P_0' u + v^{*\prime} P_0 u - v^* P_1 u =$$

$$= v^{*\prime} P_0 u + v^* P_0' u + v^* P_0 u' .$$

Exactly the same result is obtained by differentiation of the right hand side of (19) and using (8). This proves the lemma.

Definition 5: The eigenvalue problem π is said to be self-adjoint, if and only if for each pair u(t), v(t) of r-dimensional vectors belonging to $C^1[a, b]$ and satisfying the boundary conditions

$$\mathcal{B} u = \mathcal{B} v = 0 ,$$

it is the case that

$$(\mathcal{L} u, v) = (u, \mathcal{L} v). \tag{20}$$

Of special interest are the cases of self-adjoint eigenvalue problems where $\mathcal{L} = \mathcal{L}^+$. By (1) and (19) this means that

$$P_o x' + P_1 x = - (P_o^* x)' + P_1^* x = - P_o^{*\prime} x + P_1^* x - P_o^* x'$$

for every vector x. That implies

$$P_o + P_o^* = 0 , \tag{21}$$

$$P_o' = P_1 - P_1^* .$$

Substituting $\mathcal{L} = \mathcal{L}^+$ in (19) and integrating from a to b one obtains

$$(\mathcal{L} u, v) - (u, \mathcal{L} v) = v^*(b) P_o(b) u(b) - v^*(a) P_o(a) u(a) .$$

Hence if $\mathcal{L} = \mathcal{L}^+$ and if $\mathcal{B}$ is such that the boundary conditions

$$\mathcal{B} u = \mathcal{B} v = 0$$

imply

$$v^*(b)\ P_o(b)\ u(b) = v^*(a)\ P_o(a)\ u(a)\ , \tag{22}$$

then the problem π is self-adjoint. By theorem 3.2. in [3], p. 297, this is the case, if and only if

$$M\ P^{-1}(a)\ M^* = N\ P^{-1}(b)\ N^*\ .$$

The first result to be stated gives essential properties of eigenvalues and eigenfunctions of π. The question of existence of eigenvalues is treated in Chapter IV.

Theorem 1: Let the eigenvalue problem π be self-adjoint. Then the eigenvalues are real and constitute an at most denumerable set with no finite clusterpoint. Eigenfunctions corresponding to distinct eigenvalues are orthogonal.

Proof: Let $\ell = \lambda$ be an eigenvalue with $\chi(t)$ an eigenfunction of π. Then by $\mathcal{B}\,x = 0$ and (20)

$$(\mathcal{L}\chi,\ \chi) - (\chi, \mathcal{L}\ \chi) = 0$$

or equivalently

$$\lambda(\chi,\ \chi) - \bar{\lambda}(\chi,\ \chi) = (\lambda - \bar{\lambda})(\chi,\ \chi) = 0\ .$$

Since $(\chi,\ \chi) \neq 0$, $\lambda = \bar{\lambda}$ and thus λ must be real.

Let λ_i and λ_j be distinct eigenvalues with eigenfunctions $\chi_i(t)$ and $\chi_j(t)$ respectively; then, as a consequence of (20),

$$(\mathcal{L}\chi_i,\ \chi_j) - (\chi_i, \mathcal{L}\ \chi_j) = (\lambda_i - \lambda_j)(\chi_i,\ \chi_j) = 0.$$

Hence

$$(\chi_i, \chi_j) = 0.$$

This proves the orthogonality.

Let $\phi_j = \phi_j(t, \ell)$, $j = 1, 2, \ldots, r$, be the solutions of $\mathcal{L}x = \ell x$ which satisfy the initial conditions

$$\phi_{j(k)}(c, \ell) = \delta_{jk}, \qquad j, k = 1, 2, \ldots, r, \tag{23}$$

where $\phi_{j(k)}$ is the k-th component of the vector ϕ_j and c is a value in the interval [a, b]. By theorem 8.4 in [3], p. 36, the vectors $\phi_j(t, \ell)$ are entire functions in ℓ for all fixed $t \in [a, b]$, and they belong to the class C^1 for $t \in [a, b]$ and all ℓ. By (23) they are linearly independent and hence the matrix

$$\Phi(t, \ell) = \begin{pmatrix} \phi_{1(1)}(t, \ell) & \phi_{2(1)}(t, \ell) & \phi_{r(1)}(t, \ell) \\ \phi_{1(2)}(t, \ell) & \phi_{2(2)}(t, \ell) & \phi_{r(2)}(t, \ell) \\ & & \\ \phi_{1(r)}(t, \ell) & \phi_{2(r)}(t, \ell) & \phi_{r(r)}(t, \ell) \end{pmatrix} \tag{24}$$

is a fundamental matrix of the system $\mathcal{L}x = \ell x$. It is well known (cf. [3], p. 70) that every solution x(t) of the above system must be representable in the form

$$x(t, \ell) = \Phi(t, \ell)\, q ,$$

where q is a constant r-dimensional vector. Consequently ℓ is an eigenvalue, if and only if there is a vector q different from the zero vector, so that

$$\mathcal{B}\, x = \mathcal{B}\, (\Phi\ (\ell)\cdot q) = 0\ . \tag{25}$$

By defining

$$\mathcal{B}\ \Phi(\ell) = M\ \Phi(a,\ \ell) + N\ \Phi(b,\ \ell)\ , \tag{26}$$

condition (25) can be written as

$$(\ \mathcal{B}\ \Phi(\ell))\cdot q = 0, \qquad q \neq 0, \tag{27}$$

because of the associative law

$$(\ \mathcal{B}\ \Phi(\ell))\cdot q = \mathcal{B}\,(\Phi(\ell)\cdot q)\ . \tag{28}$$

Obviously there exists a q satisfying (27), if and only if the matrix $\mathcal{B}\ \Phi(\ell)$ is singular, or equivalently if and only if

$$\det\ \mathcal{B}\ \Phi(\ell) = 0\ .$$

Then and only then the linear homogeneous system of equations for the components $q_{(i)}$, $i = 1, 2, \ldots, r$, of q has a nontrivial solution. But because $\phi_{j(k)}(a,\ \ell)$ and $\phi_{j(k)}(b,\ \ell)$ for $j, k = 1, 2, \ldots, r$, are entire functions of ℓ it follows that

$$\det\ \mathcal{B}\ \Phi(\ell) = \det\ [M\ \Phi(a,\ \ell) + N\ \Phi(b,\ \ell)]$$

is also an entire function of ℓ. Because only real values of ℓ can be eigenvalues, det $\mathcal{B}\Phi(\ell)$ can only have real zeros and therefore cannot vanish identically. This implies that the zeros of det $\mathcal{B}\,\Phi(\ell)$, and hence also the eigenvalues of π, are at most countable and have no finite clusterpoint. (See e.g. [4], p. 139). This completes the proof of theorem 1.

CHAPTER III

THE GREEN'S MATRIX ASSOCIATED WITH THE EIGENVALUE PROBLEM

In this chapter the variation of constants formula (cf. [3], p. 74) will be applied. This formula states that if the r-by-r matrix $A(t)$ and the vector $b(t)$ are continuous and if $\Phi(t)$ is a fundamental matrix of the homogeneous system

$$x' = A(t)\, x, \qquad a \leq t \leq b,$$

then the vector $\phi(t)$ defined by

$$\phi(t) = \Phi(t) \int_a^t \Phi^{-1}(s)\, b(s)\, ds$$

is the unique solution on $[a, b]$ of the initial value problem

$$x' = A(t)\, x + b(t)\ ,$$

$$x(a) = 0\ .$$

Since $\Phi(t)$ is nonsingular, the integrand and hence the integral always exists. It is an obvious consequence that every solution of

$$x' = A(t)\, x + b(t)$$

can be expressed in the form

$$\phi(t) + \Phi(t)\, q \;,$$

where q is a constant vector.

The inhomogeneous problem

$$\mathcal{L}x = f + \ell x \;, \qquad a \leq t \leq b, \tag{29}$$

$$\mathcal{B}x = 0 \tag{30}$$

will now be considered, where ℓ is assumed not to be an eigenvalue and where the vector f(t) is continuous. It will be clear from the presentation that a good part of the theory does not make use of the assumption that π is self-adjoint. Substituting (1) into (29) and multiplying by $P_0^{-1}(t)$ yields the equivalent equation

$$x' = P_0^{-1}(t)\;(\ell E - P_1(t))x + P_0^{-1}(t)\; f(t) \;. \tag{31}$$

One result of the previous chapter was that the matrix $\Phi(t,\ell)$ defined in (24) is a fundamental matrix of the homogeneous equation associated with (31). Hence by the variation of constants formula every solution of (29), respectively (31) must have the form

$$u(t,\; \ell) = \int_a^t \Phi(t,\; \ell)\; \Phi^{-1}(s,\; \ell)\; P_0^{-1}(s)\; f(s)\; ds + \Phi(t,\; \ell)q, \tag{32}$$

where q is a constant vector. The problem is thus reduced to seeing whether there exists a q, so that the corresponding function u(t) also satisfies the boundary condition (30). By (2) $\mathcal{B}\, u(\ell) = 0$ means

$$M\, u(a, \ell) + N\, u(b, \ell) = 0$$

or

$$N\, \Phi(b, \ell) \int_a^b \Phi^{-1}(s, \ell)\, P_o^{-1}(s)\, f(s)\, ds + \mathcal{B}\, \Phi(\ell)\, q = 0 ,$$

where $\mathcal{B}\, \Phi(\ell)$ is the matrix defined by relation (26). Since it was assumed that ℓ was not an eigenvalue of π, det $\mathcal{B}\, \Phi(\ell) \neq 0$ and therefore one can multiply the above equation by $(\mathcal{B}\, \Phi)^{-1}$ and solve for q. The result is

$$q = -\, (\mathcal{B}\, \Phi(\ell))^{-1}\, N\Phi(b, \ell) \int_a^b \Phi^{-1}(s, \ell)\, P_o^{-1}(s)\, f(s)\, ds \tag{33}$$

for ℓ not an eigenvalue of π. One now could substitute the value of q into (32) and get an explicit formula for the solution of the problem (29, 30), provided a fundamental matrix $\Phi(t, \ell)$ is known. However, the formula becomes clearer if one introduces the matrix

$$G(t, s, \ell) = \Phi(t, \ell)\, \Phi^{-1}(s, \ell)\, P_o^{-1}(s) + \tag{34}$$

$$-\, \Phi(t, \ell)(\mathcal{B}\, \Phi(\ell))^{-1}\, N\Phi(b, \ell)\, \Phi^{-1}(s, \ell)\, P_o^{-1}(s),$$

$$(a \leq s \leq t \leq b) \cap (a < t) ,$$

and

$$G(t, s, \ell) = - \Phi(t, \ell)(\mathcal{B}\Phi(\ell))^{-1} N \Phi(b, \ell) \Phi^{-1}(s, \ell) P_o^{-1}(s),$$

$$(a \leq t < s \leq b) \cup (a = t = s).$$

By formulae (32, 33, 34) the solution of the problem (29, 30) assumes the form

$$u(t, \ell) = \int_a^b G(t, s, \ell) f(s) ds . \tag{35}$$

The matrix $G(t, s, \ell)$ defined in (34) is called the Green's matrix. This matrix may also exist for problems which are not self-adjoint. Some properties of this matrix will be listed in the following theorem.

Theorem 2: If for at least one value of ℓ the problem π has no solution except the trivial one (which is always true in the self-adjoint case), then there exists a unique r-by-r matrix $G(t, s, \ell)$ defined for all (t, s) on the square $a \leq t, s \leq b$ and for all complex ℓ except the eigenvalues of π and having the following properties:

1) $G(t, s, \ell)$ and $\frac{\partial}{\partial t} G(t, s, \ell)$ are continuous in (t, s, ℓ) for ℓ not an eigenvalue on each of the triangles $a \leq t < s \leq b$ and $a \leq s \leq t \leq b$. For fixed (t, s) these matrices are all meromorphic functions of ℓ.

2) $G(s + 0, s, \ell) - G(s - 0, s, \ell) = P_o^{-1}(s)$,

$$(a \leq s \leq b).$$

3) As a function of t, $G(t, s, \ell)$ satisfies $\mathcal{L} G = \ell G$ if $t \neq s$.

4) As a function of t, $G(t, s, \ell)$ satisfies the boundary conditions $\mathcal{B} G = 0$ for $a \leq s \leq b$.

Proof: The properties 1 to 4 follow from the definition of $G(t, s, \ell)$ in (34). Clearly $\Phi(t, \ell)$, $\frac{\partial}{\partial t} \Phi(t, \ell)$ and $P(t)$ are continuous in their variables. It is known that the inverse of a nonsingular continuous matrix is also continuous and that the sum or the product of two continuous matrices is again continuous. Hence $\Phi^{-1}(s, \ell)$ is continuous and $(\mathcal{B}\Phi(\ell))^{-1}$ is continuous for ℓ not an eigenvalue. From these facts continuity of $G(t, s, \ell)$ and $\frac{\partial G}{\partial t}(t, s, \ell)$ on the two triangles follow immediately. Analogous arguments are used to show that the matrix elements of $G(t, s, \ell)$ and of $\frac{\partial G}{\partial t}(t, s, \ell)$ can be expressed as an analytic function in ℓ divided by $\det \mathcal{B}\Phi(\ell)$. By assumption $\det \mathcal{B}\Phi(\ell)$ cannot vanish identically and it is an analytic function of ℓ; the rest of property 1 follows.

By (34)

$$G(s + 0, s, \ell) - G(s - 0, s, \ell) =$$

$$= \Phi(s + 0, \ell)\, \Phi^{-1}(s, \ell)\, P_o^{-1}(s) = P_o^{-1}(s) .$$

This proves property 2.

The proof of property 3 is straightforward, since for $t \neq s$, $G(t, s, \ell)$ can be expressed as $G(t, s, \ell) = \Phi(t, \ell)\, C(s, \ell)$ and $\mathcal{L}\, \Phi(t, \ell) = \ell\Phi(t, \ell)$.

Finally

$$\mathcal{B} G = M\, G(a, s, \ell) + N\, G(b, s, \ell) =$$

$$= - M\, \Phi(a, \ell)(\mathcal{B}\Phi(\ell))^{-1} \cdot N\Phi(b, \ell)\, \Phi^{-1}(s, \ell)\, P_o^{-1}(s) +$$

$$- N\, \Phi(b, \ell)(\mathcal{B}\Phi(\ell))^{-1}\, N\Phi(b, \ell)\, \Phi^{-1}(s, \ell)\, P_o^{-1}(s) +$$

$$+ N\, \Phi(b, \ell)\, \Phi^{-1}(s, \ell)\, P_o^{-1}(s) =$$

$$= \{ - (M\, \Phi(a, \ell) + N\, \Phi(b, \ell))(\mathcal{B}\Phi(\ell))^{-1} + E\} \cdot$$

$$\cdot\, \Phi(b, \ell)\, \Phi^{-1}(s, \ell)\, P_o^{-1}(s) = 0$$

by (26) for all values of $s \in [a, b]$. This takes care of property 4.

The last thing to be proved is that properties 1 to 4 determine the matrix $G(t, s, \ell)$ uniquely. Suppose that there were two matrices G_1 and G_2 having properties 1 to 4, then their difference

$$H(t, s, \ell) = G_1(t, s, \ell) - G_2(t, s, \ell)$$

would by properties 1 and 2 also be continuous at $t = s$ and hence on the whole square $a \leq t, s \leq b$. Property 3 would imply that H also satisfies

$$\mathcal{L} H = \ell H\,, \qquad\qquad t \neq s\,.$$

That implies that the left side and the right side limits of $\frac{\partial}{\partial t} H$ at $t = s$ exist and are equal. Hence H is in fact of class $C^1[a, b]$ and satisfies $\mathcal{L} H = \ell H$ on the whole interval $a \leq t \leq b$. H is therefore a solution matrix of $\mathcal{L} x = \ell x$ and moreover satisfies $\mathcal{B} H = 0$. But as ℓ is not an eigenvalue, there can only be trivial solutions, i.e., H consists only of zero vectors. This means $H = 0$ and the uniqueness of the Green's matrix is proved.

From now on it will be assumed that the problem π is self-adjoint and $\ell = 0$ is not an eigenvalue of π. The latter assumption is no essential restriction, since if ℓ should happen to be an eigenvalue, there exists a real constant c which is not an eigenvalue of π. Thus if

$$\mathcal{L}_1 x = \mathcal{L} x - cx ,$$

the problem

$$\pi_1: \quad \mathcal{L}_1 x = \ell x , \qquad \mathcal{B} x = 0$$

is again self-adjoint, because

$$(cu, v) = (u, cv) .$$

Moreover if λ is an eigenvalue of π_1, then $\lambda + c$ is one of π and conversely, and the eigenfunctions are the same for π and π_1. Thus if $\ell = 0$ were an eigenvalue of π, then by choice of c the following theory would hold for the problem π_1 instead of π.

Since $\ell = 0$ is not an eigenvalue of π, $G(t, s, 0)$ exists. In what follows the Green's matrix for $\ell = 0$ will be denoted by $G = G(t, s)$ and π will be assumed self-adjoint.

Corresponding to the Green's matrix $G(t, s)$ let $\mathcal{G}$ be the integral operator defined for all vectors $f \in C[a, b]$ by the identity

$$\mathcal{G} f(t) = \int_a^b G(t, s)\, f(s)\, ds \, . \tag{36}$$

$\mathcal{G} f(t)$ is again a vector which as a consequence of (35) belongs to $C^1[a, b]$ and which satisfies

$$\mathcal{L}\mathcal{G} f = f \, , \qquad \mathcal{B}\mathcal{G} f = 0 \tag{37}$$

for all continuous vectors $f(t)$. If f and g are continuous on $[a, b]$, then by (20) and (37)

$$(f, \mathcal{G} g) = (\mathcal{G} f, g) \tag{38}$$

holds. From (13) and (38) it follows immediately that $(\mathcal{G} f, f)$ is real. A further consequence of (37) is the following lemma.

<u>Lemma 4</u>: If $\ell = 0$ is not an eigenvalue of π, then a necessary and sufficient condition that π be self-adjoint is that for $t \neq s$

$$G(t, s) = G^*(s, t) \, , \qquad a \leq t, \; s \leq b \, , \tag{39}$$

holds, where G^* is the conjugate transpose of G.

<u>Proof</u>: First assume that π is self-adjoint. Then for any two vectors $f, g \in C[a, b]$ formula (38) holds. Because of the properties of the scalar product (38) can be written as

$$(\mathcal{G} f, g) = \overline{(\mathcal{G} g, f)}$$

which by (10) and (36) means

$$\int_a^b g^*(t) \left\{ \int_a^b G(t,s)\ f(s)\ ds \right\} dt = \overline{\int_a^b f^*(t) \left\{ \int_a^b G(t,s)\ g(s)\ ds \right\} dt}.$$

Application of (9) yields

$$\int_a^b \left\{ \int_a^b g^*(t)\ G(t,s)\ f(s)\ ds \right\} dt = \int_a^b \left\{ \int_a^b \overline{f^*(t)\ G(t,s)\ g(s)} \right\} dt\ .$$

Use was also made of the fact that the conjugate complex value of an integral can be obtained by integrating the conjugate complex function. Note that the integrands in the above relation are scalar functions of the two variables s and t and that they are continuous on the square $a \leq t,\ s \leq b$ except for $t = s$. Moreover they are bounded on the whole square. The expression $f^*(t)\ G(t,s)\ g(s)$ can be regarded as a one-by-one matrix, and the integrand on the right hand side as its conjugate transpose. Applying the identity

$$[f^*(t)\ G(t,\ s)\ g(s)]^* = g^*(s)\ G^*(t,\ s)\ f(t)$$

one obtains

$$\int_a^b \left\{ \int_a^b g^*(t)\ G(t,s)\ f(s)\ ds \right\} dt = \int_a^b \left\{ \int_a^b g^*(s)\ G^*(t,s)\ f(t)\ ds \right\} dt =$$

$$= \int_a^b \left\{ \int_a^b g^*(t)\ G^*(s,t)\ f(s)\ dt \right\} ds = \int_a^b \left\{ \int_a^b g^*(t)\ G^*(s,t)\ f(s)\ ds \right\} dt$$

after interchanging the dummy variables s and t and changing the order of integration, the latter step being obviously justified by the smoothness properties of the integrand. Considering that the above relation holds for all continuous vectors f and g and that G(t, s) and $G^*(s, t)$ are continuous except for t = s and applying the fundamental lemma of calculus of variations (cf. [5], p. 185) one obtains (39). Conversely, if (39) holds, (38) follows for any two continuous vectors f and g. This can be observed by reversing the above procedure. Now let u(t), v(t) be vectors belonging to C^1 on [a, b] and satisfying $\mathcal{B}u = 0$, $\mathcal{B}v = 0$, and choose $f = \mathcal{L}u$, $g = \mathcal{L}v$. Then by (29, 30, 35)

$$\mathcal{L}(u - \mathcal{G}f) = 0, \qquad \mathcal{B}(u - \mathcal{G}f) = 0,$$

$$\mathcal{L}(v - \mathcal{G}g) = 0, \qquad \mathcal{B}(v - \mathcal{G}g) = 0.$$

Since $\ell = 0$ is not an eigenvalue of π, it follows that $u = \mathcal{G}f$, $v = \mathcal{G}g$. But (38) yields

$$(f, v) = (u, g),$$

$$(\mathcal{L}u, v) = (u, \mathcal{L}v) .$$

Hence π is self-adjoint and the proof of the lemma is complete.

The operator $\mathcal{G}$ is a type of inverse to the operator $\mathcal{L}$ in the sense that

$$\mathcal{L}\mathcal{G}f = f, \qquad \mathcal{G}\mathcal{L}u = u$$

are valid for all vectors $f \in C[a, b]$ and all vectors $u \in C^1[a, b]$ satisfying $\mathcal{B}u = 0$.

CHAPTER IV

EXISTENCE OF EIGENVALUES

With G and $\mathcal{G}$ defined as before, it is clear that if λ is an eigenvalue and ϕ an eigenfunction of π corresponding to λ, then

$$\phi = \lambda \ \mathcal{G} \phi \ . \tag{40}$$

Conversely, if there exists a nontrivial $\phi \in C[a, b]$, then $\mathcal{G} \phi$ is of class C^1 and $\mathcal{L} \mathcal{G} \phi = \phi$ so that (40) implies $\mathcal{L} \phi = \lambda\phi$. Moreover $\mathcal{B} \phi = 0$, since $\mathcal{B} G = 0$.

<u>Definition 6</u>: If there exists a nontrivial $\phi \in C[a, b]$ and a complex number μ such that $\mathcal{G} \phi = \mu\phi$, then μ is said to be an eigenvalue of $\mathcal{G}$ and ϕ an eigenfunction of $\mathcal{G}$.

From above it can be concluded that the eigenfunctions of $\mathcal{G}$ are identical with those of π and the eigenvalues of $\mathcal{G}$ are reciprocals of those of π.

Whenever (38) holds $\mathcal{G}$ is said to be self-adjoint. It will be shown that such a self-adjoint operator must possess eigenvalues and in this way the result will follow for π.

<u>Definition 7</u>: An equicontinuous set of r-dimensional vectors $F = \{f\}$, where the vectors $f(t)$ are defined on a real interval $[a, b]$, is a set with the following property: Given any $\epsilon > 0$, there exists a $\delta_\epsilon > 0$ independent of f and also of $t_1, t_2 \in [a, b]$, such that

$$|f(t_2) - f(t_1)| < \epsilon$$

whenever $|t_2 - t_1| < \delta_\epsilon$, where for an r-dimensional vector

$$|f| = \sum_{i=1}^{r} |f_{(i)}| \quad . \tag{41}$$

Relation (41) defines a metric in the r-dimensional vector space. Use will be made of the inequality:

$$|f|^2 \geq \sum_{i=1}^{r} |f_{(i)}|^2 = f^* \cdot f \quad . \tag{42}$$

By a procedure exactly analogous to the one in [3], p. 5, it can be proved that for an equicontinuous set of vectors the Ascoli-lemma holds.

Lemma 5: (Ascoli). If $F = \{f\}$ is an infinite uniformly bounded equicontinuous set of vectors defined on [a, b], then F contains a sequence $\{f_n\}$, n = 1, 2, . . . , which is uniformly convergent on [a, b]. This means that there is a vector f(t) for which given any $\epsilon > 0$,

$$|f(t) - f_n(t)| < \epsilon$$

for all $t \in [a, b]$ and all $n \geq N_\epsilon$ where N_ϵ is some integer dependent on ϵ.

Lemma 6: The set of all vectors $\{\mathcal{G} u\}$ where $u \in C[a, b]$ and $\|u\| \leq 1$ is a bounded equicontinuous set of vectors.

Proof: It must be shown that for all u having the properties mentioned above

$$|\mathcal{G} u(t_2) - \mathcal{G} u(t_1)| < \epsilon ,$$

whenever $|t_2 - t_1| < \delta_\epsilon$ and $t_1, t_2 \epsilon [a, b]$ and that there exists a constant M such that

$$|\mathcal{G} u(t)| < M$$

for all $t \epsilon [a, b]$. Clearly $G(t, s)$ is bounded for $\ell = 0$ not an eigenvalue. Let

$$G(t, s) = (g_{ij}(t, s)), \qquad i, j = 1, 2, \ldots, r, \quad (43)$$

and let

$$\gamma = \sup_{\substack{s,t \epsilon [a,b] \\ \text{all } i,j}} |g_{ij}(t, s)| . \quad (44)$$

Recall that

$$\mathcal{G} u(t) = \int_a^b G(t, s) u(s) ds .$$

Hence

$$|\mathcal{G} u(t)| = \sum_{i=1}^{r} \left| \int_a^b \sum_{j=1}^{r} g_{ij}(t,s) u_{(j)}(s) ds \right| = \quad (45)$$

$$= \sum_{i=1}^{r} \left| \sum_{j=1}^{r} \int_a^b g_{ij}(t, s)\, u_{(j)}(s)\, ds \right| \leq$$

$$\leq \sum_{i=1}^{r} \left| \sum_{j=1}^{r} \gamma \int_a^b |u_{(j)}(s)| ds \right| \leq$$

$$\leq \sum_{i=1}^{r} \left(\gamma (b - a)^{1/2} \sum_{j=1}^{r} \left\{ \int_a^b |u_{(j)}(s)|^2\, ds \right\}^{1/2} \right) \leq$$

$$\leq \sum_{i=1}^{r} \gamma (b - a)^{1/2}\, r\, \|u\| = r^2\, \gamma (b - a)^{1/2}\, \|u\| \leq$$

$$\leq r^2\, \gamma (b - a)^{1/2} .$$

This proves the boundedness of $\mathcal{G}\, u(t)$. Among other facts Schwarz's inequality has been applied in the above calculations. For a fixed s, G(t, s) is continuous on [a, b] except for t = s where it has a jump. More precisely G(t, s) as a function of t is continuous on the intervals [a, s) and [s, b]. However the left side limit of G(t, s) at t = s exists also and is finite. Hence by admitting a double value at the single point t = s, which change has no effect on the integral $\mathcal{G}\, u(t)$ which is going to be considered, G(t, s) can be made continuous also on the closed interval [a, s] and hence uniformly continuous both on [a, s] and on [b, s]. This guarantees that for any $\eta > 0$ there exists a δ_1 such that

$$|g_{ij}(t_2, s) - g_{ij}(t_1, s)| < \eta$$

for all pairs (i, j) and whenever $|t_2 - t_1| < \delta_1$ and $s \notin [t_1, t_2]$ holds. Let $t_2 > t_1$ and $|t_2 - t_1| < \delta_1$, then

$$\mathcal{G}\, u(t_2) - \mathcal{G}\, u(t_1) = \int_a^b [G(t_2, s) - G(t_1, s)]\, u(s)\, ds \, .$$

$$| \mathcal{G}\, u(t_2) - \mathcal{G}\, u(t_1)| =$$

$$= \sum_{i=1}^{r} \left| \int_a^b \sum_{j=1}^{r} (g_{ij}(t_2, s) - g_{ij}(t_1, s)) u_{(j)}(s)\, ds \right| \leq$$

$$\leq \sum_{i=1}^{r} \sum_{j=1}^{r} \int_a^b |g_{ij}(t_2, s) - g_{ij}(t_1, s)|\, |u_{(j)}(s)|\, ds \, .$$

Splitting up the integral $\int_a^b$ in

$$\int_a^b = \int_a^{t_1} + \int_{t_1}^{t_2} + \int_{t_2}^{b}$$

and considering that the integrand is nonnegative one concludes

$$|\mathcal{G}u(t_2) - \mathcal{G}u(t_1)| \leq \sum_{i=1}^{r}(\eta \sum_{j=1}^{r} \int_a^b |u_{(j)}(s)|\, ds) + \tag{46}$$

$$+\sum_{i=1}^{r}(2\gamma \sum_{j=1}^{r} \int_{t_1}^{t_2} |u_{(j)}(s)|\, ds) \leq$$

$$\leq \sum_{i=1}^{r} \eta(b - a)^{1/2} \sum_{j=1}^{r} \left\{ \int_a^b |u_{(j)}|^2\, ds \right\}^{1/2} +$$

$$+ \sum_{i=1}^{r} 2\gamma(t_2 - t_1)^{1/2} \sum_{j=1}^{r} \left\{ \int_{t_1}^{t_2} |u_{(j)}|^2\, ds \right\}^{1/2} \leq$$

$$\leq r^2\, \eta(b - a)^{1/2}\, \|u\| + 2r^2\gamma(t_2 - t_1)^{1/2}\, \|u\| \leq$$

$$\leq r^2[\eta(b - a)^{1/2} + 2\gamma(t_2 - t_1)^{1/2}] .$$

Since both η and $|t_2 - t_1|$ can be made arbitrarily small, equicontinuity follows. This proves lemma 6.

Definition 8: The norm of $\mathcal{G}$ denoted by $\|\mathcal{G}\|$ is defined by

$$\|\mathcal{G}\| = \sup_{\|u\|=1} \|\mathcal{G}u\|,\ u \in C \text{ on } [a, b] .$$

One can give an upper bound of $\|\mathcal{G}\|$ remembering that (cf. formula (45))

$$\| \mathcal{G} u\|^2 = (\mathcal{G} u, \ \mathcal{G} u) = \int_a^b [\mathcal{G} u(t)]^* [\mathcal{G} u(t)] \, dt \leq$$

$$\leq \int_a^b | \mathcal{G} u(t)|^2 \, dt \leq r^4 \gamma^2 (b - a)^2 \|u\|^2 .$$

Hence

$$\| \mathcal{G} u\| \leq r^2 \gamma(b - a) \|u\|$$

and

$$\| \mathcal{G} \| \leq r^2 \gamma(b - a) < \infty .$$

Clearly for all $u \in C[a, b]$

$$\| \mathcal{G} u\| \leq \| \mathcal{G} \| \cdot \|u\|$$

holds; for, if $v = u/ \|u\|$, then $\|v\| = 1$ and $u = \|u\|v$. By linearity of $\mathcal{G}$ and the definition of $\| \mathcal{G} \|$

$$\| \mathcal{G} u\| = \|u\| \ \| \mathcal{G} v\| \leq \|u\| \ \| \mathcal{G} \| .$$

Moreover $\| \mathcal{G} \| > 0$, because $\| \mathcal{G} \|$ is clearly nonnegative and $\| \mathcal{G} \| = 0$ would imply $\mathcal{G} u \equiv 0$ for all u with $\|u\| = 1$ and $u \in C[a, b]$. It would follow that $\mathcal{L} \mathcal{G} u \equiv u \equiv 0$ and $\|u\| = 0$ what contradicts the requirement $\|u\| = 1$.

<u>Lemma 7</u>: The norm of $\mathcal{G}$ satisfies

$$\| \mathcal{G} \| = \sup_{\|u\|=1} |(\mathcal{G} u, u)|, \ u \in C[a, b].$$

<u>Proof</u>: By (38) $(\mathcal{G} u, u)$ is real. If $\|u\| = 1$, by (16)

$$|(\mathcal{G} u, u)| \leq \|\mathcal{G} u\| \, \|u\| \leq \|\mathcal{G}\|$$

and hence

$$\eta = \sup_{\|u\|=1} |(\mathcal{G} u, u)| \leq \|\mathcal{G}\| .$$

To prove the reverse inequality, consider

$$(\mathcal{G}(u + v), u + v) =$$

$$= (\mathcal{G} u, u) + (\mathcal{G} u, v) + (\mathcal{G} v, u) + (\mathcal{G} v, v).$$

From (38) $(\mathcal{G} u, v) + (\mathcal{G} v, u) = 2 \operatorname{Re}(\mathcal{G} u, v)$ follows. Moreover by definition of η and by linearity of $\mathcal{G}$

$$(\mathcal{G}(u + v), u + v) \leq \eta\|u + v\|^2 .$$

Hence

$$(\mathcal{G} u, u) + (\mathcal{G} v, v) + 2 \operatorname{Re}(\mathcal{G} u, v) \leq \eta\|u + v\|^2 .$$

Similarly considering $(\mathcal{G}(u - v), u - v)$ in the same way one obtains

$$(\mathcal{G} u, u) + (\mathcal{G} v, v) - 2 \operatorname{Re}(\mathcal{G} u, v) \geq - \eta\|u - v\|^2 .$$

Subtracting the last two equations yields

$$4 \operatorname{Re}(\mathcal{G} u, v) \leq 2\eta(\|u\|^2 + \|v\|^2) .$$

$\mathcal{G}$ u is not zero for $u \in C[a, b]$ unless $u \equiv 0$, for, if it were, $\alpha \mathcal{G} u = u$ would be zero. In particular, if $v = \mathcal{G} u / \| \mathcal{G} u \|$ in the above inequality, where $\|u\| = 1$, it follows

$$\frac{4(\mathcal{G} u, \mathcal{G} u)}{\| \mathcal{G} u \|} \leq 4\eta \ , \quad \| \mathcal{G} u \| \leq \eta$$

which completes the proof of lemma 7.

<u>Theorem 3</u>: Either $\| \mathcal{G} \|$ or $- \| \mathcal{G} \|$ is an eigenvalue for $\mathcal{G}$.

<u>Proof</u>: By lemma 7 one of the two relations

$$\| \mathcal{G} \| = \sup_{\|u\|=1} \ (\mathcal{G} u, u), \qquad u \in C[a, b],$$

$$\| \mathcal{G} \| = \sup_{\|u\|=1} \ -(\mathcal{G} u, u), \qquad u \in C[a, b]$$

holds. Suppose it is the first; then by definition of supremum there exists a sequence of vectors $u_m \in C[a,b]$, $\|u_m\| = 1$, $m = 1, 2, \ldots$, such that

$$(\mathcal{G} u_m, u_m) \to \| \mathcal{G} \| \ .$$

Let $\mu_o = \| \mathcal{G} \|$. As a subset of $\{ \mathcal{G} u \}$ by lemma 6 $\{ \mathcal{G} u_m \}$ is an equicontinuous uniformly bounded sequence. By lemma 5 there exists a subsequence, call it $\{ \mathcal{G} u_m \}$ also, which is uniformly convergent on $[a, b]$ to a continuous vector ϕ_o. It will be proved that ϕ_o is an eigenfunction with eigenvalue μ_o. By the Ascoli-lemma for every $\epsilon > 0$

$$| \mathcal{G} u_m(t) - \phi_o(t) | < \epsilon$$

for $m \geq M_\epsilon$ and all $t \in [a, b]$ or

$$\max_{a \leq t \leq b} | \mathcal{G} u_m(t) - \phi_o(t) | < \epsilon, \quad m \geq M_\epsilon .$$

It follows by (42) that

$$\| \mathcal{G} u_m - \phi_o \| \leq \left\{ \int_a^b \epsilon^2 \, dt \right\}^{1/2} = \epsilon(b - a)^{1/2}$$

for $m \geq M_\epsilon$ or

$$\| \mathcal{G} u_m - \phi_o \| \to 0 . \qquad (m \to \infty). \tag{47}$$

(47) implies also

$$\| \mathcal{G} u_m \| \to \| \phi_o \| \qquad (m \to \infty).$$

Because of $\|u_m\| = 1$ and $(\mathcal{G} u_m, u_m) \to \| \mathcal{G} \| = \mu_o$

the right side of

$$\| \mathcal{G} u_m - \mu_o u_m \|^2 = \| \mathcal{G} u_m \|^2 + \mu_o^2 \|u_m\|^2 - 2\mu_o (\mathcal{G} u_m, u_m) \tag{48}$$

tends to $\|\phi_o\| - \mu_o^2$ as $m \to \infty$. It follows that $\|\phi_o\|^2 \geq \mu_o^2 > 0$ and hence ϕ_o is not identically zero on $[a, b]$. From (48) it also follows that since $\| \mathcal{G} u_m \| \leq \mu_o^2$,

$$0 \leq \| \mathcal{G} u_m - \mu_o u_m \|^2 \leq 2 \mu_o^2 - 2 \mu_o (\mathcal{G} u_m, u_m) ,$$

which tends to zero as $m \to \infty$. Thus

$$\| \mathcal{G} u_m - \mu_o u_m \| \to 0 \qquad (m \to \infty) \ . \tag{49}$$

But by the triangle inequality

$$0 \leq \| \mathcal{G} \phi_o - \mu_o \phi_o \| \leq \| \mathcal{G} \phi_o - \mathcal{G} (\mathcal{G} u_m) \| +$$

$$+ \| \mathcal{G} (\mathcal{G} u_m) - \mu_o \mathcal{G} u_m \| + \|\mu_o \mathcal{G} u_m - \mu_o \phi_o \| \ .$$

All the three terms at the right hand side tend to zero since

$$\| \mathcal{G} \phi_o - \mathcal{G}(\mathcal{G} u_m) \| = \| \mathcal{G} (\phi_o - \mathcal{G} u_m) \| \leq \| \mathcal{G} \| \ \|\phi_o - \mathcal{G} u_m \| \to 0$$

by (47),

$$\| \mathcal{G} (\mathcal{G} u_m) - \mu_o \mathcal{G} u_m \| = \| \mathcal{G} (\mathcal{G} u_m - \mu_o u_m) \| \leq$$

$$\leq \| \mathcal{G} \| \ \| \mathcal{G} u_m - \mu_o u_m \| \to 0$$

by (49), and

$$\|\mu_o \mathcal{G} u_m - \mu_o \phi_o \| = \mu_o \ \| \mathcal{G} u_m - \phi_o \| \to 0$$

by (47). This implies $\| \mathcal{G} \phi_o - \mu_o \phi_o \| = 0$, thus proving

$$\mathcal{G} \phi_o = \mu_o \phi_o \ ,$$

and completes the proof of the theorem in case $\| \mathcal{G} \| = \sup\limits_{\|u\|=1} (\mathcal{G} u, u)$.

An analogous proof applies in case $\| \mathcal{G} \| = \sup\limits_{\|u\|=1} - (\mathcal{G} u, u)$.

Let $\chi_o = \phi_o / \|\phi_o\|$. Then $\|\chi_o\| = 1$ and χ_o is said to be a normalized eigenfunction. Let

$$G_1(t, s) = G(t, s) - \mu_o \chi_o(t) \chi_o^*(s),$$

i.e., $G_1(t, s)$ is the matrix with elements

$$g^1_{ik}(t, s) = g_{ik}(t, s) - \mu_o \chi_{o(i)}(t) \cdot \bar{\chi}_{o(k)}(s)$$

where $\chi_{o(i)}(t)$ is the i-th component of $\chi_o(t)$. Then define the operator $\mathcal{G}_1$ for $u \in C[a, b]$ by

$$\mathcal{G}_1 u(t) = \int_a^b G_1(t, s)\, u(s)\, ds = \mathcal{G} u(t) - \mu_o(u, \chi_o)\, \chi_o(t) .$$

Then $\mathcal{G}_1$ has the same properties as $\mathcal{G}$ was shown to have in lemmas 6 and 7. In particular if $\|\mathcal{G}_1\| \neq 0$ and

$$\sup |(\mathcal{G}_1 u, u)| = |\mu_1|$$

where $u \in C[a, b]$, $\|u\| = 1$ and μ_1 real, then μ_1 is an eigenvalue for $\mathcal{G}_1$ and there exists a nontrivial $\phi_1 \in C[a, b]$ such that $\mathcal{G}_1 \phi_1 = \mu_1 \phi_1$. Let $\chi_1 = \phi_1 / \|\phi_1\|$. Note that for any $u \in C[a, b]$

$$(\mathcal{G}_1 u, \chi_o) = 0$$

since

$$(\mathcal{G}_1 u, \chi_o) = (\mathcal{G} u, \chi_o) - \mu_o(u, \chi_o) (\chi_o, \chi_o) =$$

$$= (u, \mathcal{G} \chi_o) - \mu_o(u, \chi_o) = (u, \mu_o \chi_o) - \mu_o(u, \chi_o) = 0$$

and specially

$$(\mathcal{G}_1 \chi_1, \chi_o) = (\mu_1 \chi_1, \chi_o) = \mu_1(\chi_1, \chi_o) = 0,$$

i.e., χ_1 is orthogonal to χ_o. Therefore

$$\mathcal{G}\chi_1 = \mathcal{G}_1 \chi_1 + \mu_o \chi_o(\chi_1, \chi_o) = \mathcal{G}_1 \chi_1 = \mu_o \chi_1$$

and hence χ_1 is an eigenfunction of $\mathcal{G}$. From the extremal property, $|\mu_1| \leq |\mu_o|$. Letting

$$G_2(t, s) = G_1(t, s) - \mu_1 \chi_1(t) \chi_1^*(s)$$

and proceeding as before one establishes the existence of χ_2 and μ_2 with $|\mu_2| \leq |\mu_1|$ and χ_2 orthogonal to χ_1 and χ_o. In this way the existence of an orthonormal sequence $\{\chi_k\}$, $k = 0, 1, 2, \ldots$, is established, orthonormal in the sense that

$$(\chi_1, \chi_k) = \delta_{ik}, \qquad i, k = 0, 1, 2, \ldots$$

This process can terminate only if, for some m, $\| \mathcal{G}_m \| = 0$. But for any vector f of class C[a, b]

$$\mathcal{L} \mathcal{G}_m f = f - \sum_{j=1}^{m-1} \mu_j(f, \chi_j) \mathcal{L} \chi_j .$$

With $\| \mathcal{G}_m \| = 0$, this implies

$$f = \sum_{j=0}^{m-1} (f, \chi_j)\, \chi_j \ . \tag{50}$$

Since the χ_j are all of class C^1 and f can be taken to be in class C but not in C^1, (50) cannot hold. Thus $\| \mathcal{G}_m \| > 0$ for all m, and there is therefore a countably infinite number of eigenvalues and eigenfunctions for every self-adjoint operator $\mathcal{L}$.

CHAPTER V

EXPANSION AND COMPLETENESS THEOREM

Lemma 8: (Bessel's inequality). If a vector $f \in (L_2[a, b])^r$ (cf. Chapter I) and if $\{\chi_k\}$ is an orthonormal sequence of eigenfunctions for the self-adjoint problem π, then the series

$$\sum_{k=0}^{\infty} |(f, \chi_k)|^2$$

is convergent and

$$\sum_{k=0}^{\infty} |(f, \chi_k)|^2 \leq \|f\|^2 .$$

(Note that by assumption $\|f\| = (f, f)^{1/2}$ exists and is $< \infty$.)

Proof: For any finite $m > 0$

$$0 \leq \| f - \sum_{k=0}^{m} (f, \chi_k) \chi_k \|^2 = (f, f) +$$

$$- (f, \sum_{k=0}^{m} (f, \chi_k) \chi_k) - (\sum_{k=0}^{m} (f, \chi_k) \chi_k, f) +$$

$$+ \left(\sum_{k=0}^{m} (f, \chi_k) \chi_k, \sum_{\ell=0}^{m} (f, \chi_\ell) \chi_\ell \right) =$$

$$= (f, f) - \sum_{k=0}^{m} \overline{(f, \chi_k)} (f, \chi_k) - \sum_{k=0}^{m} (f, \chi_k) (\chi_k, f) +$$

$$+ \sum_{k=0}^{m} \sum_{\ell=0}^{m} (f, \chi_k) \overline{(f, \chi_\ell)} \delta_{k\ell} =$$

$$= \|f\|^2 - \sum_{k=0}^{m} |(f, \chi_k)|^2 .$$

Theorem 4: Let the vector f belong to $C^1[a, b]$ and satisfy the boundary conditions $\mathcal{B} f = 0$. Then on $a \leq t \leq b$,

$$f = \sum_{k=0}^{\infty} (f, \chi_k) \chi_k \tag{51}$$

where the series converges uniformly on $[a, b]$.

Remark: By multiplying (51) by f^* on the left and integrating from a to b one gets

$$\|f\|^2 = \sum_{k=0}^{\infty} (f, \chi_k) (\chi_k, f) = \sum_{k=0}^{\infty} |(f, \chi_k)|^2 . \tag{52}$$

That means that the system $\{\chi_k\}$ is complete in the space of vectorfunctions f belonging to C^1 and satisfying $\mathcal{B} f = 0$.

Proof: Considering the conjugate transposed form of the relation

$$\int_a^b G(t,\ s)\ \chi_k(s)\ ds = \mu_k\ \chi_k(t)\ , \tag{53}$$

using the trivial identity

$$[\int_a^b A(s)\ ds]^* = \int_a^b A^*(s)\ ds$$

which holds for all integrable matrices $A(s)$ and can be verified by computing the single matrix elements, one obtains the equation

$$\int_a^b \chi_k^*(s)\ G^*(t,\ s)\ ds = \mu_k\ \chi_k^*(t)\ . \tag{54}$$

In the sequel some properties of the trace of a square matrix will be needed. If C is an r-by-r matrix, then by definition

$$\operatorname{Tr} C = \sum_{i=1}^{r} c_{ii}\ .$$

Hence if $C = AB$ where A and B are r-by-r matrices,

$$\operatorname{Tr} C = \operatorname{Tr} AB = \sum_{i=1}^{r}\ \sum_{j=1}^{r} a_{ij}\ b_{ji}\ . \tag{55}$$

Note that

$$\mathrm{Tr}\ AB = \mathrm{Tr}\ BA\ . \tag{56}$$

Special cases of (55) will now be considered.

$$\text{a)}\quad \mathrm{Tr}\ A^* A = \sum_{i=1}^{r} \sum_{j=1}^{r} a_{ij}{}^* a_{ji} = \sum_{i,j=1}^{r} \bar{a}_{ji}\, a_{ji} = \tag{57}$$

$$= \sum_{i,j=1}^{r} |a_{ji}|^2 \geq 0$$

Therefore, for any square matrix A, Tr A^* A is real and nonnegative.

b) Let x and y be two r-dimensional vectors and let $B = xy^*$. Then by (55) and $b_{ij} = x_i\, \overline{y_j}$,

$$\mathrm{Tr}\ A\ x\ y^* = \sum_{i=1}^{r} \sum_{j=1}^{r} a_{ij}\, x_j\, \overline{y_i} = \tag{58}$$

$$= \sum_{i=1}^{r} \overline{y_i} \sum_{j=1}^{r} a_{ij}\, x_j = y^* A\ x\ .$$

c) by (56, 58)

$$\mathrm{Tr}\ x\ y^*\ A = y^*\ A\ x\ . \tag{59}$$

Finally, if A_i, i = 1, 2, . . . , n, are r-by-r matrices, then

$$\mathrm{Tr}\left(\sum_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathrm{Tr}\, A_i \;. \tag{60}$$

These properties having been derived the proof can now be continued. By (57) for each $m \geq 0$ and all $s,\ t \in [a,\ b]$

$$\mathrm{Tr}\,\{G^*_{m+1}(t,\ s) \cdot G_{m+1}(t,\ s)\} \geq 0$$

and is real. Since

$$G_{m+1}(t,\ s) = G(t,\ s) - \sum_{i=0}^{m} \mu_i\ \chi_i(t)\ \chi_i^*(s)\ ,$$

the above inequality yields

$$0 \leq \mathrm{Tr}\,\{[G^*(t,\ s) - \sum_{i=0}^{m} \mu_i\ \chi_i(s)\ \chi_i^*(t)][G(t,\ s) +$$

$$- \sum_{j=0}^{m} \mu_j\ \chi_j(t)\ \chi_j^*(s)]\} =$$

$$= \mathrm{Tr}\,\{G^*(t,\ s)\ G(t,\ s)\} +$$

$$- \sum_{j=0}^{m} \mu_j\ \mathrm{Tr}\,\{G^*(t,\ s)\ \chi_j(t)\ \chi_j^*(s)\} +$$

$$- \sum_{i=0}^{m} \mu_i \operatorname{Tr} \{\chi_i(s) \chi_i^*(t) G(t, s)\} +$$

$$+ \sum_{i=0}^{m} \sum_{j=0}^{m} \mu_i \mu_j \operatorname{Tr} \{\chi_i(s) \chi_i^*(t) \chi_j(t) \chi_j^*(s)\} =$$

$$= \operatorname{Tr} \{G^*(t, s) G(t, s) - \sum_{j=0}^{m} \mu_j \chi_j^*(s) G^*(t, s) \chi_j(t) +$$

$$- \sum_{i=0}^{m} \mu_i \chi_i^*(t) G(t, s) \chi_i(s) +$$

$$+ \sum_{i=0}^{m} \sum_{j=0}^{m} \mu_i \mu_j \chi_i^*(t) \chi_j(t) \chi_j^*(s) \chi_i(s) \geq 0 .$$

The last inequality will now be integrated with respect to s from a to b. Considering (53, 54) and the orthogonality properties of the χ_i one obtains

$$\int_a^b \operatorname{Tr} \{G^*(t,s) G(t,s)\} ds - \sum_{j=0}^{m} \mu_j^2 \chi_j^*(t) \chi_j(t) +$$

$$- \sum_{i=0}^{m} \mu_i^2 \chi_i^*(t) \chi_i(t) + \sum_{i=0}^{m} \sum_{j=0}^{m} \mu_i \mu_j \chi_i^*(t) \chi_j(t) \cdot \delta_{ij} =$$

$$= \int_a^b \mathrm{Tr}\ \{G^*(t,s)\ G(t,s)\}\ ds - \sum_{j=0}^{m} \mu_j^{\ 2}\ \chi_j^*(t)\ \chi_j(t) \geq 0\ .$$

Integrating also with respect to t yields

$$\int_a^b \int_a^b \mathrm{Tr}\ \{G^*(t,s)\ G(t,s)\}\ ds\ dt \geq \sum_{j=0}^{m} \mu_j^{\ 2} \tag{61}$$

for all integers $m \geq 0$. Because of

$$\mathrm{Tr}\ \{G^*(t,s)\ G(t,s)\} = \sum_{j=1}^{r} \sum_{i=1}^{r} |g_{ij}(t,s)|^2 \leq r^2\ \gamma^2$$

(61) implies, letting m tend to ∞,

$$\sum_{j=0}^{\infty} \mu_j^{\ 2} \leq (b - a)^2\ r^2\ \gamma^2 < \infty$$

where γ is defined by (44). Hence $|\mu_j| \to 0$ as $j \to \infty$.

Consider for any integer $m \geq 1$

$$G_m(t,s) = G(t,s) - \sum_{k=0}^{m-1} \mu_k\ \chi_k(t)\ \chi_k^{\ *}(s)\ .$$

From the extremal properties it is known that $\| \mathcal{G}_m \| = |\mu_m|$. Thus for any vector $u \in C[a,\ b]$

$$\| \mathcal{G}_m u\| = \| \mathcal{G} u - \sum_{k=0}^{m-1} \mu_k(u, \chi_k)\chi_k\| \leq |\mu_m| \; \|u\|$$

or, since $|\mu_m| \to 0$ as $m \to \infty$,

$$\lim_{m \to \infty} \| \mathcal{G} u - \sum_{k=0}^{m-1} \mu_k(u, \chi_k) \chi_k\| = 0. \tag{62}$$

For any $q > p$

$$\sum_{k=p}^{q} \mu_k(u, \chi_k)\chi_k = \mathcal{G} \left(\sum_{k=p}^{q} (u, \chi_k)\chi_k \right) .$$

Since $| \mathcal{G} u| \leq r^2 \gamma(b - a)^{1/2} \|u\|$, by (45) it follows that

$$\left| \sum_{k=p}^{q} \mu_k(u, \chi_k)\chi_k \right| \leq r^2 \gamma(b - a)^{1/2} \left(\sum_{k=p}^{q} |(u, \chi_k)|^2 \right)^{1/2} .$$

By Bessel's inequality the last sum tends to zero as $p, q \to \infty$. Thus

$$\sum_{k=0}^{\infty} \mu_k(u, \chi_k)\chi_k$$

is uniformly convergent on [a, b], and therefore represents a continuous function there. Since $\mathcal{G} u$ is also continuous, (62) implies that

$$\mathcal{G}\, u(t) = \sum_{k=0}^{\infty} \mu_k(u,\ \chi_k)\ \chi_k(t)\ . \tag{63}$$

Given any $f \in C^1$ satisfying $\mathcal{B} f = 0$, then $u = \mathcal{L} f \in C[a,\ b]$ and $f = \mathcal{G}\, u$. Therefore

$$f = \sum_{k=0}^{\infty} \mu_k(\mathcal{L}\, f,\ \chi_k)\ \chi_k =$$

$$= \sum_{k=0}^{\infty} \mu_k(f,\ \mathcal{L}\ \chi_k)\ \chi_k =$$

$$= \sum_{k=0}^{\infty} \mu_k\ \lambda_k(f,\ \chi_k)\ \chi_k =$$

$$= \sum_{k=0}^{\infty} (f,\ \chi_k)\ \chi_k\ .$$

This completes the proof of theorem 4.

<u>Theorem 5</u>: If the vector $f \in (L_2[a,\ b])^r$, then

$$f = \sum_{k=0}^{\infty} (f,\ \chi_k)\ \chi_k$$

in the mean, i.e.,

$$\lim_{m \to \infty} \|f - \sum_{k=0}^{m} (f, \chi_k) \chi_k\| = 0 . \tag{64}$$

Further, Parseval's equality holds:

$$\|f\|^2 = \sum_{k=0}^{\infty} |(f, \chi_k)|^2.$$

Proof: In order to prove theorem 5, one may use the following lemma:

Lemma 9: The class of vectors x belonging to C^1 on [a, b] and satisfying $\mathcal{B}x = 0$ is everywhere dense in $(L_2[a, b])^r$, i.e., for every $f \epsilon (L_2[a,b])^r$ and every $\epsilon > 0$, there exists an $\tilde{f} \epsilon C^1[a, b]$ with $\mathcal{B}\tilde{f} = 0$, such that

$$\|f - \tilde{f}\| < \epsilon .$$

Proof of lemma 9: Consider the set S of all scalar complex functions u belonging to $C^1[a, b]$. Then the subset $\tilde{S} \subset S$ containing all the functions $u \epsilon C^1[a, b]$, for which $u(a) = 0$, $u(b) = 0$, is everywhere dense in S in the sense that for every $u \epsilon S$ and every $\epsilon > 0$, there exists a $\tilde{u} \epsilon \tilde{S}$ such that

$$\int_a^b |u - \tilde{u}|^2 \, dt < \epsilon.$$

Indeed, for some $\delta > 0$, $\delta < \frac{b - a}{2}$, define

$$\tilde{u} = u, \qquad a + \delta \leq t \leq b - \delta ,$$

$$\tilde{u} = \frac{t - a}{\delta^2} [\delta (t - a + \delta) \cdot u'(a + \delta) + (a + 2\delta - t) \cdot u(a + \delta)],$$

$$, \ a \leq t \leq a + \delta ,$$

$$\tilde{u} = \frac{b - t}{\delta^2} [\delta (t - b + \delta) \cdot u'(b - \delta) + (t + 2\delta - b) \cdot u(b - \delta)],$$

$$, \ b - \delta \leq t \leq b;$$

then clearly $\tilde{u} \in \tilde{S}$. Since $u'(t)$ and $u(t)$ are bounded on $[a, b]$, there exists a positive number M, so that

$$|u'(t)| \leq M, \qquad a \leq t \leq b,$$

$$|u(t)| \leq M, \qquad a \leq t \leq b.$$

With $\tilde{u}$ defined as above

$$\int_a^b |u - \tilde{u}|^2 \, dt = \int_a^{a+\delta} |u - \tilde{u}|^2 \, dt + \int_{b-\delta}^b |u - \tilde{u}|^2 \, dt.$$

But on $[a, a + \delta]$

$$|\tilde{u}| \leq \frac{|t - a|}{\delta^2} \Big\{ \delta |u'(a + \delta)| \cdot |t - (a + \delta)| +$$

$$+ |u(a + \delta)| \cdot |(a + 2\delta) - t| \Big\} \leq \frac{1}{\delta} \Big\{ \delta^2 M + 2\delta M \Big\} \leq 3M$$

provided that $\delta \leq 1$. Hence

$$|u - \tilde{u}| \leq 4M$$

$$\int_a^{a+\delta} |u - \tilde{u}|^2 \, dt \leq 16 \, \delta \, M^2.$$

In the same way one shows that

$$\int_{b-\delta}^{b} |u - \tilde{u}|^2 \, dt \leq 16 \, \delta \, M^2,$$

and hence

$$\int_a^b |u - \tilde{u}|^2 \, dt \leq 32 \, \delta \, M^2 < \epsilon$$

if

$$0 < \delta \leq \min(1, \frac{b - a}{2}, \frac{\epsilon}{32 \, M^2}) \, .$$

This shows that $\tilde{S}$ is everywhere dense in S and also in the set of all polynomials defined on [a, b] which is a subset of S. But since the polynomials are everywhere dense in $L_2[a, b]$, the set $\tilde{S}$ is also everywhere dense in $L_2[a, b]$ (cf. [6], p. 72).

By definition of the norm

$$\|f - \tilde{f}\| = \int_a^b (f - \tilde{f})^*(f - \tilde{f}) \, dt =$$

$$= \sum_{i=1}^{r} \int_a^b |f_{(i)} - \tilde{f}_{(i)}|^2 \, dt \; .$$

$f \epsilon (L_2[a, b])^r$ implies $f_{(i)} \epsilon L_2[a, b]$, for $i = 1, 2, \ldots, r$, and hence the above result can be applied to state that for every $f_{(i)}$ and every $\epsilon^* > 0$, there exists an $\tilde{f}_{(i)} \epsilon \tilde{S}$, so that

$$\int_a^b |f_{(i)} - \tilde{f}_{(i)}|^2 \, dt \leq \epsilon^* .$$

Clearly the vector $\tilde{f}$ having components $\tilde{f}_{(i)}$, $i = 1, \ldots, r$, belongs to $C^1[a, b]$ and because all $\tilde{f}_{(i)}$ vanish at a and at b, $\mathcal{B}\tilde{f} = 0$. Moreover, if $\epsilon^* < \frac{\epsilon}{n}$,

$$\|f - \tilde{f}\| \leq n \, \epsilon^* < \epsilon \; .$$

This proves lemma 9.

Now the proof of theorem 5 can also be completed. By the properties of the norm

$$\|f - \sum_{k=0}^{m} (f, \chi_k) \chi_k \| \leq \|f - \tilde{f}\| +$$

$$+ \|\tilde{f} - \sum_{k=0}^{m} (\tilde{f}, \chi_k) \chi_k\| + \| \sum_{k=0}^{m} (\tilde{f} - f, \chi_k) \chi_k\| \; .$$

By lemma 9 and by theorem 5, for any $\epsilon > 0$, there exists an $\tilde{f} \epsilon C^1[a, b]$ satisfying $\mathcal{B}\tilde{f} = 0$ and an integer M so that

$$\|\tilde{f} - f\| < \epsilon ,$$

$$\|\tilde{f} - \sum_{k=0}^{m} (\tilde{f}, \chi_k)\ \chi_k\| < \epsilon , \ m > M .$$

Also, by orthogonality of the χ_k

$$\|\sum_{k=0}^{m} (\tilde{f} - f, \chi_k)\ \chi_k\| = \left\{ \sum_{k=0}^{m} (\tilde{f} - f, \chi_k)(\chi_k, \tilde{f} - f)(\chi_k, \chi_k)\right\}^{1/2} =$$

$$= \left\{\sum_{k=0}^{m} |(\tilde{f} - f, \chi_k)|^2 \right\}^{1/2} .$$

By Bessel's inequality this is again less than or equal to

$$\|\tilde{f} - f\| < \epsilon .$$

The final conclusion

$$\|f - \sum_{k=0}^{m} (f, \chi_k)\ \chi_k\| < 3\epsilon , \ m > M,$$

follows, proving (64). Parseval's equality is implied directly from (64), since

$$\left\|f - \sum_{k=0}^{m} (f, \chi_k)\, \chi_k\right\|^2 = \|f\|^2 - \sum_{k=0}^{m} |(f, \chi_k)|^2 .$$

CHAPTER VI

EXAMPLES

Two examples of self-adjoint eigenvalue problems on $[0, \pi]$ will be given. The number of equations r will be equal to two and the operator $\mathcal{L}$ will be defined by the matrices

$$P_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} . \tag{66}$$

It is easy to verify that $\mathcal{L} = \mathcal{L}^+$ (formula (21)).

In the first example let

$$N = M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} . \tag{67}$$

As a simple calculation shows, two vectors u(t), v(t), satisfying $\mathcal{B} x = 0$ with $\mathcal{B}$ given by formulae (2, 67) also satisfy (22). Since (21) also holds, the problem is self-adjoint. Expressed by means of scalar equations the eigenvalue problem reads:

$$x'_{(2)} = \ell\, x_{(1)} , \qquad x_{(1)}(0) + x_{(1)}(\pi) = 0 , \tag{68}$$

$$- x'_{(1)} = \ell\, x_{(2)} , \qquad x_{(2)}(0) + x_{(2)}(\pi) = 0 .$$

It easily follows that any solution x of the system of differential equations has the form

$$x_{(1)}(t) = \alpha \cos \ell\, t + \beta \sin \ell\, t \ ,$$

$$x_{(2)}(t) = - \beta \cos \ell\, t + \alpha \sin \ell\, t \ ,$$

where α and β are arbitrary complex constants. The boundary conditions require

$$\alpha(1 + \cos \pi \ell) + \beta \sin \pi \ell = 0 \ , \tag{69}$$

$$\alpha \sin \pi \ell - \beta (1 + \cos \pi \ell) = 0 \ .$$

In order that (68) have a nontrivial solution, the above homogeneous linear system for α and β must have a nontrivial solution. Hence its determinant must vanish. This means that

$$(1 + \cos \pi \ell)^2 + \sin^2 \pi \ell = 0 \ .$$

This is equivalent to

$$\cos \pi \ell = - 1,$$

for which the solution is

$$\ell = 2n - 1, \qquad n = 0, \pm 1, \pm 2, \ldots$$

Hence the eigenvalues are

$$\lambda_{2n-1} = 2n - 1, \qquad n = 0, \pm 1, \pm 2, \ldots$$

(In this example it is more convenient to number the eigenvalues from $-\infty$ to $+\infty$. Of course there exists no difficulty to renumber them from 0 to ∞ in order to achieve the same notation as in Chapters IV and V.)

Note that for ℓ an eigenvalue the matrix of the system (69) is the zero matrix which is of rank zero. This suggests the occurrence of double eigenvalues at $\ell = \lambda_{2n-1}$, i.e., the existence of two linearly independent solutions of (69). Indeed, both

$$\chi_{2n-1}(t) = \frac{1}{\pi^{1/2}} \begin{pmatrix} \cos (2n - 1) t \\ \sin (2n - 1) t \end{pmatrix} \tag{70}$$

and

$$\chi_{2n}(t) = \frac{1}{\pi^{1/2}} \begin{pmatrix} \sin (2n - 1) t \\ - \cos (2n - 1) t \end{pmatrix}$$

$$n = 0, \pm 1, \ldots$$

satisfy (68) for $\ell = 2n - 1$. In addition $\chi_{2n-1}(t)$ and $\chi_{2n}(t)$ are linearly independent and

$$(\chi_i, \chi_j) = \delta_{ij}\,, \qquad i = 2n-1,\ 2n, \qquad j = 2n-1,\ 2n\ .$$

Together with the results in Chapters IV and V, it can be concluded that the vectors

$$\chi_n(t), \quad n = 0, \pm 1, \pm 2, \ldots$$

form a complete orthonormal system of vectors in the space $L_2[0, \pi] \times L_2[0, \pi]$. Incidentally, this result can be verified directly by using the properties of Fourier series. It can be said that the eigenvalues of the problem are

$$\lambda_{2n-1} = \lambda_{2n} = 2n - 1, \qquad n = 0, \pm 1, \pm 2, \ldots \tag{71}$$

For ℓ not an eigenvalue, the Green's matrix was calculated to be equal to

$$G(s,t,\ell) = \frac{1 + 2\cos \ell\pi}{2(1 + \cos \ell\pi)} \begin{pmatrix} \sin(s-t)\ell & -\cos(s-t)\ell \\ \cos(s-t)\ell & \sin(s-t)\ell \end{pmatrix} + \tag{72}$$

$$- \frac{1}{2(1 + \cos \ell\pi)} \begin{pmatrix} \sin(s-\pi-t)\ell & -\cos(s-\pi-t)\ell \\ \cos(s-\pi-t)\ell & \sin(s-\pi-t)\ell \end{pmatrix},$$

$$(0 \leq s \leq t \leq \pi) \cap (t \neq 0),$$

$$G(s,t,\ell) = -\frac{1}{2(1 + \cos \ell\pi)} \begin{pmatrix} \sin(s-t)\ell & -\cos(s-t)\ell \\ \cos(s-t)\ell & \sin(s-t)\ell \end{pmatrix} +$$

$$- \frac{1}{2(1 + \cos \ell\pi)} \begin{pmatrix} \sin(s-\pi-t)\ell & -\cos(s-\pi-t)\ell \\ \cos(s-\pi-t)\ell & \sin(s-\pi-t)\ell \end{pmatrix},$$

$$(0 \leq t < s \leq \pi) \cup (s = t = 0).$$

The second example deals with the same operator but with a different set of boundary conditions, namely

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad N = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{73}$$

An easy calculation shows that also this problem is self-adjoint. The scalar differential equations and boundary conditions defining the problem are

$$x'_{(2)} = \ell\, x_{(1)}\,, \qquad x_{(1)}(0) = 0\,, \tag{74}$$

$$-\,x'_{(1)} = \ell\, x_{(2)}\,, \qquad x_{(2)}(\pi) = 0\,.$$

Again any solution of the differential equation only has the form

$$x_{(1)} = \alpha \cos \ell t + \beta \sin \ell t\,,$$

$$x_{(2)} = -\,\beta \cos \ell t + \alpha \sin \ell t$$

with arbitrary complex α, β. The boundary conditions imply

$$\alpha = 0, \qquad \cos \ell \pi = 0\,,$$

i.e., $\ell = n + \frac{1}{2}$, $n = 0, \pm 1, \pm 2, \ldots$ for a nontrivial solution to exist. Hence the eigenvalues which in this example are simple are

$$\lambda_n = n + \frac{1}{2}\,, \qquad n = 0, \pm 1, \pm 2, \ldots \tag{75}$$

and the orthonormal set of eigenfunctions is given by

$$\chi_n(t) = \frac{1}{\pi^{1/2}} \begin{pmatrix} \sin(n + \frac{1}{2})t \\ -\cos(n + \frac{1}{2})t \end{pmatrix}, \quad n = 0, \pm 1, \ldots \tag{76}$$

Finally the Green's matrix is given by

$$\frac{1}{\cos \ell \pi} \begin{pmatrix} \sin s \ell \cos(t-\pi)\ell & -\cos s \ell \cos(t-\pi)\ell \\ \sin s \ell \sin(t-\pi)\ell & -\cos s \ell \sin(t-\pi)\ell \end{pmatrix}, \tag{77}$$

$$(0 \leq s \leq t \leq \pi) \cap (t \neq 0),$$

$$\frac{1}{\cos \ell \pi} \begin{pmatrix} \sin t \ell \cos(s-\pi)\ell & \sin t \ell \sin(s-\pi)\ell \\ -\cos t \ell \cos(s-\pi)\ell & -\cos t \ell \sin(s-\pi)\ell \end{pmatrix},$$

$$(0 \leq t < s \leq \pi) \cup (0 = t = s).$$

for ℓ not an eigenvalue, i.e., $\cos \ell \pi \neq 0$.

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